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ON SUPERSYMMETRIC QUANTUM MECHANICS

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Abstract. This paper constitutes a review on $\mathcal{N} = 2$ fractional supersymmetric Quantum Mechanics of order k. The presentation is based on the introduction of a generalized Weyl-Heisenberg algebra W_k. It is shown how a general Hamiltonian can be associated with the algebra W_k. This general Hamiltonian covers various supersymmetrical versions of dynamical systems (Morse system, Pöschl-Teller system, fractional supersymmetric oscillator of order k, etc.). The case of ordinary supersymmetric Quantum Mechanics corresponds to $k = 2$. A connection between fractional supersymmetric Quantum Mechanics and ordinary supersymmetric Quantum Mechanics is briefly described. A realization of the algebra W_k, of the $\mathcal{N} = 2$ supercharges and of the corresponding Hamiltonian is given in terms of deformed-bosons and k-fermions as well as in terms of differential operators.

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1 Introducing supersymmetry

Supersymmetry (SUperSYmmetry or SUSY) can be defined as a symmetry between bosons and fermions (as considered as elementary particles or simply as degrees of freedom). In other words, SUSY is based on the postulated existence of operators Q_α which transform a bosonic field into a fermionic field and *vice versa*. In the context of quantum mechanics, such symmetry operators induce a Z_2-grading of the Hilbert space of quantum states. In a more general way, fractional SUSY corresponds to a Z_k-grading for which the Hilbert space involves both bosonic degrees of freedom (associated with bosons) and k-fermionic degrees of freedom (associated with k-fermions to be described below) with $k \in \mathbf{N} \setminus \{0,1,2\}$; the case $k=2$ corresponds to ordinary SUSY.

The concept of SUSY is very useful in Quantum Physics and Quantum Chemistry. It was first introduced in elementary particle physics on the basis of the unification of internal and external symmetries [1,2]. More precisely, SUSY goes back to the sixtees when many attemps were done in order to unify *external symmetries* (described by the Poincaré group) and *internal symmetries* (described by gauge groups) *for elementary particles*. These attempts led to a no-go theorem by Coleman and Mandula in 1967 [1] which states that, under reasonable assumptions concerning the S-matrix, the unification of internal and external symmetries can be achieved solely through the introduction of the direct product of the Poincaré group with the relevant gauge group. In conclusion, this unification brings nothing new since it simply amounts to consider separately the two kinds of symmetries. A way to escape this no-go theorem was proposed by Haag, Lopuszanski and Sohnius in 1975 [2]: The remedy consists in replacing the Poincaré group by an extended Poincaré group, the Poincaré *supergroup* (or Z_2-*graded* Poincaré *group*).

Let us briefly discuss how to introduce the Poincaré supergroup. We know that the Poincaré group has ten generators (six $M_{\mu\nu} \equiv -M_{\nu\mu}$ and four P_μ with $\mu, \nu \in \{0, 1, 2, 3\}$): Three $M_{\mu\nu}$ describe ordinary rotations, three $M_{\mu\nu}$ describe special Lorentz transformations and the four P_μ describe space-time translations. The Lie algebra of the Poincaré group is characterized by the commutation relations

$$[M_{\mu\nu}, M_{\rho\sigma}] = -\mathrm{i}(g_{\mu\rho}M_{\nu\sigma} + g_{\nu\sigma}M_{\mu\rho} - g_{\mu\sigma}M_{\nu\rho} - g_{\nu\rho}M_{\mu\sigma})$$

$$[M_{\mu\nu}, P_\lambda] = -\mathrm{i}(g_{\mu\lambda}P_\nu - g_{\nu\lambda}P_\mu), \quad [P_\mu, P_\nu] = 0$$

where $(g_{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1)$. The minimal extension of the Poincaré group into a Poincaré supergroup gives rise to a Lie *superalgebra* (or Z_2-*graded* Lie *algebra*) involving the ten generators $M_{\mu\nu}$ and P_μ plus four new generators Q_α (with $\alpha \in \{1, 2, 3, 4\}$) referred to as supercharges. The Lie superalgebra of the Poincaré supergroup is then described by the commutation relations above plus the additional commutation relations

$$[M_{\mu\nu}, Q_\alpha] = \frac{\mathrm{i}}{4}\left([\gamma_\mu, \gamma_\nu]\right)_{\alpha\beta} Q_\beta, \quad [P_\mu, Q_\alpha] = 0$$

and the anticommutation relations

$$\{Q_\alpha, Q_\beta\} = -2\left(\gamma^\mu C\right)_{\alpha\beta} P_\mu$$

where the γ's are Dirac matrices, C the charge conjugation matrix and $\alpha, \beta \in \{1, 2, 3, 4\}$.

What is the consequence of this increase of symmetries (i.e., passing from 10 to 14 generators) and of the introduction of anticommutators ? In general, an increase of symmetry yields an increase of degeneracies. For instance, in condensed matter physics, passing from the tetragonal symmetry to the cubical symmetry leads, from a situation where the degrees of degeneracy are 1 and 2, to a situation where the degrees of degeneracy are 1, 2 and 3 (in the absence of accidental degeneracies). Another possible consequence of the increase of symmetry is the occurrence of new states or new particles. For instance, the number of particles is doubled when going from the Schrödinger equation (with Galilean invariance) to the Dirac equation (with Lorentz invariance): With each particle of mass m, spin S and charge e is associated an antiparticle of mass m, spin S and charge $-e$. In a similar way, when passing from the Poincaré group to the Poincaré supergroup, we associate with a known particle of mass m, spin S and charge e a new particle of mass m', spin $S' = \left|S \pm \frac{1}{2}\right|$ and charge e. The particle and the new particle, called a *sparticle* or a *particlino* (according to whether as the known particle is a fermion or a boson), are accommodated in a given irreducible representation of the Z_2-graded Poincaré group [3]. Consequently, they should have the same mass. However, we have $m' \neq m$ because SUSY is a broken symmetry. In terms of field theory, the sparticle or particlino field results from the action of a supercharge Q_α on the particle field (and *vice versa*):

$$Q_\alpha : \text{fermion} \mapsto \text{sfermion} = \text{boson}$$

$$Q_\alpha : \text{boson} \mapsto \text{bosino} = \text{fermion}$$

(see also Refs. 3 and 4). We thus speak of a selectron (a particle of spin 0 associated with the electron) and of a photino (a particle of spin $\frac{1}{2}$ associated with the photon).

The experimental evidence for SUSY is not yet firmly established. Some arguments in favour of SUSY come from: (i) condensed matter physics with the fractional quantum Hall effect and high temperature superconductivity [5]; (ii) nuclear physics where SUSY could connect the complex structure of odd-odd nuclei to much simpler even-even and odd-A systems [6]; and (iii) high energy physics (especially in the search of supersymmetric particles and the lighter Higgs boson) where the observed signal around 115 GeV/c^2 for a neutral Higgs boson is compatible with the hypotheses of SUSY [7].

It is not our intention to further discuss SUSY from the viewpoint of condensed matter physics, nuclear physics and elementary particle physics. We shall rather focus our attention on supersymmetric Quantum Mechanics (sQM), a supersymmetric quantum field theory in $D = 1 + 0$ dimension [8] that has received a great deal of attention in the last twenty years and is still in a state of development. In recent years, the investigation of quantum groups, with deformation parameters taken as roots of unity, has been a catalyst for the study of fractional sQM which is an extension of ordinary sQM. This extension takes its motivation in the so-called intermediate or exotic statistics like: (i) anyonic statistics in $D = 1+2$ dimensions connected to braid groups [5,9-11], (ii) para-bosonic and para-fermionic statistics in $D = 1 + 3$ dimensions connected to permutation groups [12-15], and (iii) q-deformed statistics (see, for instance, [16,17]) arising from q-deformed oscillator algebras [18-23]. Along this vein, intermediate statistics constitute a useful tool for the study of physical phenomena in condensed matter physics (e.g., fractional quantum Hall effect and supraconductivity at high critical temperature).

Ordinary sQM needs two degrees of freedom: one bosonic degree (described by a complex variable) and one fermionic degree (described by a Grassmann variable).

From a mathematical point of view, we then have a Z_2-grading of the Hilbert space of physical states (involving bosonic and fermionic states). Fractional sQM of order k is an extension of ordinary sQM for which the Z_2-grading is replaced by a Z_k-grading with $k \in \mathbf{N} \setminus \{0, 1, 2\}$. The Z_k-grading corresponds to a bosonic degree of freedom (described again by a complex variable) and a para-fermionic or k-fermionic degree of freedom (described by a generalized Grassmann variable of order k). In other words, to pass from ordinary supersymmetry or ordinary sQM to fractional supersymmetry or fractional sQM of order k, we retain the bosonic variable and replace the fermionic variable by a para-fermionic or k-fermionic variable (which can be represented by a $k \times k$ matrix).

A possible approach to fractional sQM of order k thus amounts to replace fermions by para-fermions of order $k-1$. This yields para-supersymmetric Quantum Mechanics as first developed, with one boson and one para-fermion of order 2, by Rubakov and Spiridonov [24] and extended by various authors [25-30]. An alternative approach to fractional sQM of order k consists in replacing fermions by k-fermions which are objects interpolating between bosons (for $k \to \infty$) and fermions (for $k = 2$) and which satisfy a generalized Pauli exclusion principle according to which one cannot put more than $k-1$ particles on a given quantum state [31]. The k-fermions proved to be useful in describing Bose-Einstein condensation in low dimensions [16] (see also Ref. [17]). They take their origin in a pair of q- and $\bar{q}$-oscillator algebras (or q- and $\bar{q}$-uon algebras) with

$$q = \frac{1}{\bar{q}} = \exp\left(\frac{2\pi \mathrm{i}}{k}\right), \tag{1}$$

where $k \in \mathbf{N} \setminus \{0, 1\}$ [31,32]. Along this line, a fractional supersymmetric oscillator was derived in terms of boson and k-fermion operators in Ref. [33].

Fractional sQM can be developed also without an explicit introduction of k-fermionic degrees of freedom [34,35]. In this respect, fractional sQM of order $k = 3$ was worked out [35] owing to the introduction of a C_λ-extended oscillator algebra in the framework of an extension of the construction of sQM with one bosonic degree of freedom only [34].

The connection between fractional sQM (and thus ordinary sQM) and quantum groups has been worked out by several authors [36-44] mainly with applications to exotic statistics in mind. In particular, LeClair and Vafa [36] studied the isomorphism between the affine quantum algebra $U_q(sl_2)$ and $\mathcal{N} = 2$ fractional sQM in $D = 1 + 1$ dimensions when q^2 goes to a root of unity ($\mathcal{N}$ is the number of supercharges); in the special case where $q^2 \to -1$, they recovered ordinary sQM.

This paper is a review on fractional sQM. Ordinary sQM and fractional sQM can be useful in Quantum Chemistry from two points of view. Firstly, SUSY allows to deal with dynamical systems exhibiting fermionic (possibly k-fermionic) and bosonic degrees of freedom. Secondly, it makes possible to treat in a unified way nonrelativistic systems controled by potentials connected via transformations similar to the ones used in the factorization method. It is thus hoped that the the present work will open an avenue of future investigations in the applications of SUSY to Quantum Chemistry.

The content of this paper is as follows. Fractional sQM of order k is approached from a generalized Weyl-Heisenberg algebra W_k (defined in Section 2). In Section 3, a general fractional supersymmetric Hamiltonian is derived from the generators of W_k. This Hamiltonian is specialized (in Section 4) to the case of a fractional supersymmetric oscillator. Finally, differential realizations, involving bosonic and generalized Grassmannian variables, of fractional sQM are given in Section 5 for some particular cases of W_k. Some concluding remarks (in Section 6) close this paper. Two appendices complete this paper: The quantum algebra $U_q(sl_2)$ with $q^k = 1$ is connected to W_k in Appendix A and a boson + k-fermion decomposition of a Q-uon for $Q \to q$ is derived in Appendix B.

Throughout the present work, we use the notation $[A, B]_Q = AB - QBA$ for any complex number Q and any pair of operators A and B. As particular cases, we have $[A, B] \equiv [A, B]_- = [A, B]_1$ and $\{A, B\} \equiv [A, B]_+ = [A, B]_{-1}$ for the commutator and the anticommutator, respectively, of A and B. As usual, $\bar{z}$ denotes the complex conjugate of the number z and $A^\dagger$ stands for the Hermitean conjugate of the operator A.

2 A generalized Weyl-Heisenberg algebra W_k

2.1 The algebra W_k

For fixed k, with $k \in \mathbf{N} \setminus \{0, 1\}$, we define a generalized Weyl-Heisenberg algebra, denoted as W_k, as an algebra spanned by four linear operators X_- (annihilation operator), X_+ (creation operator), N (number operator) and K (Z_k-grading operator) acting on some separable Hilbert space and satisfying the following relations:

$$[X_-, X_+] = \sum_{s=0}^{k-1} f_s(N)\, \Pi_s, \tag{2a}$$

$$[N, X_-] = -X_-, \quad [N, X_+] = +X_+, \tag{2b}$$

$$[K, X_+]_q = [K, X_-]_{\bar{q}} = 0, \tag{2c}$$

$$[K, N] = 0, \tag{2d}$$

$$K^k = 1, \tag{2e}$$

where q is the k-th root of unity given by (1). In Eq. (2a), the f_s are arbitrary functions (see below) and the operators Π_s are polynomials in K defined by

$$\Pi_s = \frac{1}{k} \sum_{t=0}^{k-1} q^{-st}\, K^t \tag{3}$$

for $s = 0, 1, \cdots, k-1$. Furthermore, we suppose that the operator K is unitary ($K^\dagger = K^{-1}$), the operator N is self-adjoint ($N^\dagger = N$), and the operators X_-

and X_+ are connected via Hermitean conjugation ($X_-^\dagger = X_+$). The functions $f_s : N \mapsto f_s(N)$ must satisfy the constrain relation

$$f_s(N)^\dagger = f_s(N)$$

(with $s = 0, 1, \cdots, k-1$) in order that $X_+ = X_-^\dagger$ be verified.

2.2 Projection operators for W_k

It is easy to show that we have the resolution of the identity operator

$$\sum_{s=0}^{k-1} \Pi_s = 1$$

and the idempotency relation

$$\Pi_s \Pi_t = \delta(s,t) \Pi_s$$

where δ is the Kronecker symbol. Consequently, the k Hermitean operators Π_s are projection operators for the cyclic group $Z_k = \{1, K, \cdots, K^{k-1}\}$ of order k spanned by the generator K. In addition, these projection operators satisfy

$$\Pi_s X_+ = X_+ \Pi_{s-1} \Leftrightarrow X_- \Pi_s = \Pi_{s-1} X_- \tag{4}$$

with the convention $\Pi_{-1} \equiv \Pi_{k-1}$ and $\Pi_k \equiv \Pi_0$ (more generally, $\Pi_{s+kn} \equiv \Pi_s$ for $n \in \mathbf{Z}$). Note that Eq. (3) can be reversed in the form

$$K^t = \sum_{s=0}^{k-1} q^{ts} \, \Pi_s$$

with $t = 0, 1, \cdots, k-1$.

The projection operators Π_s can also be discussed in a form closer to the one generally used in Quantum Chemistry. The eigenvalues of K are q^t with $t = 0, 1, \cdots, k-1$. One could easily establish that the eigenvector ϕ_t associated with q^t satisfies

$$\Pi_s \phi_t = \delta(s,t) \phi_t,$$

a relation that makes clearer the definition of the projection operators Π_s.

2.3 Representation of W_k

We now consider an Hilbertean representation of the algebra W_k. Let $\mathcal{F}$ be the Hilbert-Fock space on which the generators of W_k act. Since K obeys the cyclicity condition $K^k = 1$, the operator K admits the set $\{1, q, \cdots, q^{k-1}\}$ of eigenvalues. It thus makes it possible to endow the representation space $\mathcal{F}$ of the algebra W_k with a Z_k-grading as

$$\mathcal{F} = \bigoplus_{s=0}^{k-1} \mathcal{F}_s \tag{5a}$$

where the subspace

$$\mathcal{F}_s = \{|n, s\rangle : n = 1, 2, \cdots, d\}, \tag{5b}$$

with

$$K|n, s\rangle = q^s|n, s\rangle,$$

is a d-dimensional space (d can be finite or infinite). Therefore, to each eigenvalue q^s (with $s = 0, 1, \cdots, k-1$) we associate a subspace $\mathcal{F}_s$ of $\mathcal{F}$. It is evident that

$$\Pi_s|n, t\rangle = \delta(s, t)\,|n, s\rangle$$

and, thus, the application $\Pi_s : \mathcal{F} \to \mathcal{F}_s$ yields a projection of $\mathcal{F}$ onto its subspace $\mathcal{F}_s$.

The action of $X_\pm$ and N on $\mathcal{F}$ can be taken to be

$$N|n, s\rangle = n|n, s\rangle$$

and

$$X_-|n, s\rangle = \sqrt{F_s(n)}\,|n-1, s-1\rangle, \quad s \neq 0, \tag{6a}$$

$$X_-|n, 0\rangle = \sqrt{F_s(n)}\,|n-1, k-1\rangle, \quad s = 0, \tag{6b}$$

$$X_+|n, s\rangle = \sqrt{F_{s+1}(n+1)}\,|n+1, s+1\rangle, \quad s \neq k-1, \tag{6c}$$

$$X_+|n, k-1\rangle = \sqrt{F_{s+1}(n+1)}\,|n+1, 0\rangle, \quad s = k-1. \tag{6d}$$

The function F_s is a structure function that fulfills

$$F_{s+1}(n+1) - F_s(n) = f_s(n)$$

with the initial condition $F_s(0) = 0$ for $s = 0, 1, \cdots, k-1$ (cf. Refs. [45,46]).

2.4 A deformed-boson + k-fermion realization of W_k

2.4.1 The realization of W_k

In Section 2.4, the main tools consist of k pairs $(b(s)_-, b(s)_+)$ with $s = 0, 1, \cdots, k-1$ of deformed-bosons and one pair (f_-, f_+) of k-fermions. The operators $f_\pm$ satisfy (see Appendix B)

$$[f_-, f_+]_q = 1, \quad f_-^k = f_+^k = 0,$$

and the operators $b(s)_\pm$ the commutation relation

$$[b(s)_-, b(s)_+] = f_s(N), \tag{7}$$

where the functions f_s with $s = 0, 1, \cdots, k-1$ and the operator N occur in Eq. (2). In addition, the pairs (f_-, f_+) and $(b(s)_-, b(s)_+)$ are two pairs of commuting operators and the operators $b(s)_\pm$ commute with the projection operators Π_t with $s, t = 0, 1, \cdots, k-1$. Of course, we have $b(s)_+ = b(s)_-^\dagger$ but $f_+ \neq f_-^\dagger$ except for $k = 2$. The k-fermions introduced in [31] and recently discussed in [32] are objects interpolating between fermions and bosons (the case $k = 2$ corresponds to ordinary fermions and the case $k \to \infty$ to ordinary bosons); the k-fermions also share some features of the anyons introduced in [9-11]. We now introduce the linear combinations

$$b_- = \sum_{s=0}^{k-1} b(s)_- \, \Pi_s, \quad b_+ = \sum_{s=0}^{k-1} b(s)_+ \, \Pi_s.$$

It is immediate to verify that we have the commutation relation

$$[b_-, b_+] = \sum_{s=0}^{k-1} f_s(N) \, \Pi_s, \tag{8}$$

a companion of Eq. (7). Indeed, in the case where $f_s = 1$ ($s = 0, 1, \cdots, k-1$), the two pairs (b_-, b_+) and (f_-, f_+) may be considered as originating from a pair (a_-, a_+) of Q-uons through the Q-uon $\to$ boson + k-fermion decomposition described in Appendix B.

We are now in a situation to find a realization of the generators X_-, X_+ and K of the algebra W_k in terms of the b's and f's. Let us define the shift operators X_- and X_+ by (see [33])

$$X_- = b_- \left(f_- + \frac{f_+^{k-1}}{[[k-1]]_q!} \right), \tag{9}$$

$$X_+ = b_+ \left(f_- + \frac{f_+^{k-1}}{[[k-1]]_q!} \right)^{k-1}, \tag{10}$$

where the q-deformed factorial is given by

$$[[n]]_q! = [[1]]_q \, [[2]]_q \, \cdots \, [[n]]_q$$

for $n \in \mathbf{N}^*$ (and $[[0]]_q! = 1$) and where the symbol $[[\ ]]_q$ is defined by

$$[[X]]_q = \frac{1 - q^X}{1 - q}$$

with X an arbitrary operator or number. It is also always possible to find a representation for which the relation $X_-^\dagger = X_+$ holds (see Section 2.4.2). Furthermore, we define the grading operator K by

$$K = [f_-, f_+]. \tag{11}$$

In view of the remarkable property

$$\left(f_- + \frac{f_+^{k-1}}{[[k-1]]_q!} \right)^k = 1,$$

we obtain

$$[X_-, X_+] = [b_-, b_+]. \tag{12}$$

Equations (8) and (12) show that Eq. (2a) is satisfied. It can be checked also that the operators X_-, X_+ and K satisfy Eqs. (2c) and (2e). Of course, Eqs. (2b) and (2d) have to be considered as postulates. However, note that the operator N is formally given in terms of the b's by Eq. (7). We thus have a realization of the generalized Weyl-Heisenberg algebra W_k by multilinear forms involving k pairs $(b(s)_-, b(s)_+)$ of deformed-boson operators ($s = 0, 1, \cdots, k-1$) and one pair (f_-, f_+) of k-fermion operators.

2.4.2 Actions on the space $\mathcal{F}$

Equation (7) is satisfied by

$$b(s)_- b(s)_+ = F_{s+1}(N+1), \quad b(s)_+ b(s)_- = F_s(N),$$

where the structure functions F_s are connected to the structure constants f_s via

$$F_{s+1}(N+1) - F_s(N) = f_s(N).$$

Let us consider the operators X_- and X_+ defined by Eqs. (9) and (10) and acting on the Hilbert-Fock space $\mathcal{F}$ (see Eq. (5)). We choose the action of the constituent operators $b_\pm$ and $f_\pm$ on the state $|n, s\rangle$ to be given by

$$b_-|n, s\rangle = b(s)_-|n, s\rangle = \sqrt{F_s(n + \sigma - \frac{1}{2})}\, |n-1, s\rangle,$$

$$b_+|n, s\rangle = b(s)_+|n, s\rangle = \sqrt{F_s(n + \sigma + \frac{1}{2})}\, |n+1, s\rangle,$$

and

$$f_-|n, s\rangle = |n, s-1\rangle, \quad f_-|n, 0\rangle = 0,$$

$$f_+|n, s\rangle = [[s+1]]_q\, |n, s+1\rangle, \quad f_+|n, k-1\rangle = 0,$$

where $\sigma = \frac{1}{2}$, $n \in \mathbf{N}$ and $s = 0, 1, \cdots, k-1$. The action of $b_\pm$ is standard and the action of $f_\pm$ corresponds to $\alpha = 0$ and $\beta = 1$ in Appendix B. Then, we can show that the relationships (6) are satisfied. In this representation, it is easy to prove that the Hermitean conjugation relation $X_-^\dagger = X_+$ is true.

2.5 Particular cases for W_k

The algebra W_k covers a great number of situations encountered in the literature (see Refs. [33-35,47,48]). These situations differ by the form given to the right-hand side of (2a) and can be classified as follows.

(i) As a particular case, the algebra W_2 for $k = 2$ with

$$[X_-, X_+] = 1 + c\, K, \quad [N, X_\pm] = \pm X_\pm,$$

$$[K, X_\pm]_+ = 0, \quad [K, N] = 0, \quad K^2 = 1,$$

where c is a real constant ($f_0 = 1 + c$, $f_1 = 1 - c$), corresponds to the Calogero-Vasiliev [47] algebra considered in Ref. [48] for describing a system of two anyons, with an $\mathrm{Sl}(2, \mathbf{R})$ dynamical symmetry, subjected to an intense magnetic field and in Ref. [34] for constructing sQM without fermions. Of course, for $k = 2$ and $c = 0$ we recover the algebra describing the ordinary or Z_2-graded supersymmetric oscillator.

If we define

$$c_s = \frac{1}{k} \sum_{t=0}^{k-1} q^{-ts} \, f_t(N), \tag{13}$$

with the functions f_t chosen in such a way that c_s is independent of N (for $s = 0, 1, \cdots, k-1$), the algebra W_k defined by

$$[X_-, X_+] = \sum_{s=0}^{k-1} c_s \, K^s, \tag{14}$$

together with Eqs. (2b)-(2e), corresponds to the C_λ-extented harmonic oscillator algebra introduced in Ref. [35] for formulating fractional sQM of order $k = 3$.

(ii) Going back to the general case where $k \in \mathbf{N} \backslash \{0, 1\}$, if we assume in Eq. (2a) that $f_s = G$ is independent of s with $G(N)^\dagger = G(N)$, we get

$$[X_-, X_+] = G(N). \tag{15}$$

We refer the algebra W_k defined by Eq. (15) together with Eqs. (2b)-(2e) to as a nonlinear Weyl-Heisenberg algebra (see also Ref. [15]). The latter algebra was considered by the authors as a generalization of the Z_k-graded Weyl-Heisenberg algebra describing a generalized fractional supersymmetric oscillator [33].

(iii) As a particular case, for $G = 1$ we have

$$[X_-, X_+] = 1 \tag{16}$$

and here we can take

$$N = X_+ X_-. \tag{17}$$

The algebra W_k defined by Eqs. (16) and (17) together with Eqs. (2b)-(2e) corresponds to the Z_k-graded Weyl-Heisenberg algebra connected to the fractional supersymmetric oscillator studied in Ref. [33].

(iv) Finally, it is to be noted that the affine quantum algebra $U_q(sl_2)$ with $q^k = 1$ can be considered as a special case of the generalized Weyl-Heisenberg algebra W_k (see Appendix A). This result is valid for all the representations (studied in Ref. [49]) of the algebra $U_q(sl_2)$.

3 A general supersymmetric Hamiltonian

3.1 Axiomatic of supersymmetry

The axiomatic of *ordinary* sQM is known since more than 20 years. A doublet of linear operators (H, Q), where H is a self-adjoint operator and where the operators H and Q act on a separable Hilbert space and satisfy the relations

$$Q_- = Q, \quad Q_+ = Q^\dagger \quad (\Rightarrow\ Q_-^\dagger = Q_+), \quad Q_\pm^2 = 0$$

$$Q_-Q_+ + Q_+Q_- = H$$

$$[H, Q_\pm] = 0$$

is said to define a supersymmetric quantum-mechanical system (see Ref. [8]). The operator H is referred to as the Hamiltonian of the system spanned by the supersymmetry operator Q. The latter operator yields the two nilpotent operators, of order $k = 2$, Q_- and Q_+. These dependent operators are called supercharge operators. The system described by the doublet (H, Q) is called an ordinary supersymmetric quantum-mechanical system ; it corresponds to a Z_2-grading with fermionic and bosonic states.

The preceding definition of ordinary sQM can be extended to *fractional* sQM of order k, with $k \in \mathbf{N} \setminus \{0, 1, 2\}$. Following Refs. [24]-[28], a doublet of linear operators $(H, Q)_k$, with H a self-adjoint operator and Q a supersymmetry operator, acting on a separable Hilbert space and satisfying the relations

$$Q_- = Q, \quad Q_+ = Q^\dagger \quad (\Rightarrow\ Q_-^\dagger = Q_+), \quad Q_\pm^k = 0 \tag{18a}$$

$$Q_-^{k-1}Q_+ + Q_-^{k-2}Q_+Q_- + \cdots + Q_+Q_-^{k-1} = Q_-^{k-2}H \tag{18b}$$

$$[H, Q_\pm] = 0 \tag{18c}$$

is said to define a k-fractional supersymmetric quantum-mechanical system. The operator H is the Hamiltonian of the system spanned by the two (dependent) supercharge operators Q_- and Q_+ that are nilpotent operators of order k. This system corresponds to a Z_k-grading with k-fermionic and bosonic states. It is clear that the special case $k = 2$ corresponds to an ordinary supersymmetric quantum-mechanical system. Note that the definition (18) corresponds to a $\mathcal{N} = 2$ formulation of fractional sQM of order k ($\frac{1}{2}\mathcal{N}$ is the number of independent supercharges).

3.2 Supercharges

It is possible to associate a supersymmetry operator Q with a generalized Weyl-Heisenberg algebra W_k. We define the supercharge operators Q_- and Q_+ by

$$Q_- = X_-(1 - \Pi_1) = (1 - \Pi_0)X_-, \tag{19a}$$

$$Q_+ = X_+(1 - \Pi_0) = (1 - \Pi_1)X_+, \tag{19b}$$

or alternatively

$$Q_- = X_-(\Pi_2 + \cdots + \Pi_{k-2} + \Pi_{k-1} + \Pi_0), \tag{19c}$$

$$Q_+ = X_+(\Pi_1 + \Pi_2 + \cdots + \Pi_{k-2} + \Pi_{k-1}). \tag{19d}$$

Indeed, we have here one of k, with $k \in \mathbf{N} \setminus \{0, 1\}$, possible equivalent definitions of the supercharges Q_- and Q_+ corresponding to the k circular permutations of the indices $0, 1, \cdots, k-1$. Obviously, we have the Hermitean conjugation relation

$$Q_-^\dagger = Q_+.$$

By making use of the commutation relations between the projection operators Π_s and the shift operators X_- and X_+ [see Eq. (4)], we easily get

$$Q_-^m = X_-^m(\Pi_0 + \Pi_{m+1} + \Pi_{m+2} + \cdots + \Pi_{k-1}) \tag{20a}$$

$$Q_+^m = X_+^m(\Pi_1 + \Pi_2 + \cdots + \Pi_{k-m-1} + \Pi_{k-m}) \tag{20b}$$

for $m = 0, 1, \cdots, k-1$. By combining Eqs. (19) and (20), we obtain

$$Q_-^k = Q_+^k = 0.$$

Hence, the supercharge operators Q_- and Q_+ are nilpotent operators of order k.

We continue with some relations of central importance for the derivation of a supersymmetric Hamiltonian. The basic relations are

$$Q_+Q_-^m = X_+X_-^m(1 - \Pi_m)(\Pi_0 + \Pi_{m+1} + \cdots + \Pi_{k-1}) \tag{21a}$$

$$Q_-^mQ_+ = X_-^mX_+(1 - \Pi_0)(\Pi_m + \Pi_{m+1} + \cdots + \Pi_{k-1}) \tag{21b}$$

with $m = 0, 1, \cdots, k-1$. From Eq. (21), we can derive the following identities giving $Q_-^mQ_+Q_-^\ell$ with $m + \ell = k - 1$.

(i) We have

$$Q_+Q_-^{k-1} = X_+X_-^{k-1}\Pi_0 \tag{22a}$$

$$Q_-^{k-1}Q_+ = X_-^{k-1}X_+\Pi_{k-1} \tag{22b}$$

in the limiting cases corresponding to $(m = 0, \ell = k-1)$ and $(m = k-1, \ell = 0)$.

(ii) Furthermore, we have

$$Q_-^mQ_+Q_-^\ell = X_-^mX_+X_-^\ell(\Pi_0 + \Pi_{k-1}) \tag{22c}$$

with the conditions $(m \neq 0, \ell \neq k-1)$ and $(m \neq k-1, \ell \neq 0)$.

3.3 The general Hamiltonian

We are now in a position to associate a k-fractional supersymmetric quantum-mechanical system with the algebra W_k characterized by a given set of functions $\{f_s : s = 0, 1, \cdots, k-1\}$. By using Eqs. (2), (19) and (22), we find that the most general expression of H defined by Eq. (18) is [50]

$$H = (k-1)X_+X_- - \sum_{s=3}^{k}\sum_{t=2}^{s-1}(t-1)\, f_t(N-s+t)\Pi_s$$

$$-\sum_{s=1}^{k-1}\sum_{t=s}^{k-1}(t-k)\, f_t(N-s+t)\Pi_s \tag{23}$$

in terms of the product X_+X_-, the operators Π_s and the functions f_s. In the general case, we can check that

$$H^\dagger = H \tag{24}$$

and

$$[H, Q_-] = [H, Q_+] = 0. \tag{25}$$

Equations (24) and (25) show that the two supercharge operators Q_- and Q_+ are two (non independent) constants of the motion for the Hamiltonian system described by the self-adjoint operator H. As a result, the doublet $(H, Q)_k$ associated to W_k satisfies Eq. (18) and thus defines a k-fractional supersymmetric quantum-mechanical system. From Eqs. (23)-(25), it can be seen that the Hamiltonian H is a linear combination of the projection operators Π_s with coefficients corresponding to isospectral Hamiltonians (or supersymmetric partners) associated with the various subspaces $\mathcal{F}_s$ with $s = 0, 1, \cdots, k-1$ (see Section 3.5).

The Hamiltonian H and the supercharges Q_- and Q_+ can be expressed by means of the deformed-bosons and k-fermions. By using the identity

$$\Pi_s \left(f_- + \frac{f_+^{k-1}}{[[k-1]]_q!} \right)^n = \left(f_- + \frac{f_+^{k-1}}{[[k-1]]_q!} \right)^n \Pi_{s+n},$$

with $s = 0, 1, \cdots, k-1$ and $n \in \mathbf{N}$, the supercharges Q_- and Q_+ can be rewritten as

$$Q_- = \left(f_- + \frac{f_+^{k-1}}{[[k-1]]_q!} \right) \sum_{s=1}^{k-1} b(s)_-\, \Pi_{s+1},$$

$$Q_+ = \left(f_- + \frac{f_+^{k-1}}{[[k-1]]_q!} \right)^{k-1} \sum_{s=1}^{k-1} b(s+1)_+\, \Pi_s,$$

with the convention $b(k)_+ = b(0)_+$. Then, the supersymmetric Hamiltonian H given by Eq. (23) assumes a form involving the operators $b(s)_\pm$, the projection operators Π_s (that may be written with k-fermion operators), and the structure constants f_s with $s = 0, 1, \cdots, k-1$.

3.4 Particular cases for the Hamiltonian

The extended Weyl-Heisenberg algebra W_k covers numerous algebras (see Section 2.5). Therefore, the general expression (23) for the Hamiltonian H associated with W_k can be particularized to some interesting cases describing exactly solvable one-dimensional systems. Indeed, the particular system corresponding to a given set $\{f_s : s = 0, 1, \cdots, k-1\}$ yields, in a Schrödinger picture, a particular dynamical system with a specific potential.

(i) In the particular case $k = 2$, by taking $f_0 = 1 + c$ and $f_1 = 1 - c$, where c is a real constant, the Hamiltonian (23) gives back the one derived in Ref. [34].

More generally, by restricting the functions f_t in Eq. (23) to constants (independent of N) defined by

$$f_s = \sum_{t=0}^{k-1} q^{st} \, c_t$$

in terms of the constants c_t (cf. Eq. (13)), the so-obtained Hamiltonian H corresponds to the C_λ-oscillator fully investigated for $k = 3$ in Ref. [35]. The case

$$\forall s \in \{0, 1, \cdots, k-1\} \ : \ f_s(N) = f_s \text{ independent of } N$$

corresponds to systems with cyclic shape-invariant potentials (in the sense of Ref. [51]).

(ii) In the case $f_s = G$ (independent of $s = 0, 1, \cdots, k-1$), i.e., for a generalized Weyl-Heisenberg algebra W_k defined by (2b)-(2e) and (15), the Hamiltonian H can be written as

$$H = (k-1)X_+X_- - \sum_{s=2}^{k-1}\sum_{t=1}^{s-1} G(N-t)(1 - \Pi_1 - \Pi_2 - \cdots - \Pi_s)$$
$$+ \sum_{s=1}^{k-1}\sum_{t=0}^{k-s-1} (k-s-t)G(N+t)\,\Pi_s. \qquad (26)$$

The latter expression was derived in Ref. [33].

(iii) The case

$$G(N) = aN + b \text{ where } (a, b) \in \mathbf{R}^2$$

corresponds to systems with translational shape-invariant potentials (in the sense of Ref. [52]). For instance, the case $(a = 0, b > 0)$ corresponds to the harmonic oscillator potential, the case $(a < 0, b > 0)$ to the Morse potential and the case $(a > 0, b > 0)$ to the Pöschl-Teller potential. For these various potentials, the part of W_k spanned by X_-, X_+ and N can be identified with the ordinary Weyl-Heisenberg algebra for $(a = 0, b \neq 0)$, with the su(2) Lie algebra for $(a < 0, b > 0)$ and with the su(1,1) Lie algebra for $(a > 0, b > 0)$.

(iv) If $G = 1$, i.e., for a Weyl-Heisenberg algebra defined by (2b)-(2e) and (16), Eq. (26) leads to the Hamiltonian

$$H = (k-1)X_+X_- + (k-1)\sum_{s=0}^{k-1}(s + 1 - \frac{1}{2}k)\Pi_{k-s} \qquad (27)$$

for a fractional supersymmetric oscillator. The energy spectrum of H is made of equally spaced levels with a ground state (singlet), a first excited state (doublet), a second excited state (triplet), $\cdots$, a $(k-2)$-th excited state ($(k-1)$-plet) followed by an infinite sequence of further excited states (all k-plets), see Section 4.3.

(v) In the case where the algebra W_k is restricted to $U_q(sl_2)$, see Appendix A, the corresponding Hamiltonian H is given by Eq. (23) where the f_t are given in Appendix A. This yields

$$H = (k-1)J_+J_- + \frac{1}{\sin\frac{2\pi}{k}} \sum_{s=3}^{k} \sum_{t=2}^{s-1} (t-1)\, \sin\frac{4\pi t}{k} \Pi_s$$

$$+ \frac{1}{\sin\frac{2\pi}{k}} \sum_{s=1}^{k-1} \sum_{t=s}^{k-1} (t-k)\, \sin\frac{4\pi t}{k} \Pi_s. \tag{28}$$

Alternatively, Eq. (28) can be rewritten in the form (26) where $X_\pm \equiv J_\pm$ and $N \equiv J_3$ and where the function G is defined by

$$G(X) = -[2X]_q,$$

where the symbol $[\ ]_q$ is defined by

$$[2X]_q = \frac{q^{2X} - q^{-2X}}{q - q^{-1}}$$

with X an arbitrary operator or number. The quadratic term J_+J_- can be expressed in term of the Casimir operator J^2 of $U_q(sl_2)$, see Appendix A. Thus, the so-obtained expression for the Hamiltonian H is a simple function of J^2 and J_3.

3.5 A connection between fractional sQM and ordinary sQM

In order to establish a connection between *fractional* sQM of order k and *ordinary* sQM (corresponding to $k = 2$), it is necessary to construct subsystems from the doublet $(H, Q)_k$ that correspond to ordinary supersymmetric quantum-mechanical systems. This may be achieved in the following way [50]. Equation (23) can be rewritten as

$$H = \sum_{s=1}^{k} H_s\, \Pi_s \tag{29}$$

where

$$H_s \equiv H_s(N) = (k-1)F(N) - \sum_{t=2}^{k-1}(t-1)f_t(N-s+t) + (k-1)\sum_{t=s}^{k-1} f_t(N-s+t). \tag{30}$$

It can be shown that the operators $H_k \equiv H_0, H_{k-1}, \cdots, H_1$, turn out to be isospectral operators. By introducing

$$X(s)_- = \sum_n [H_s(n)]^{\frac{1}{2}} |n-1, s-1\rangle\langle n, s|$$

$$X(s)_+ = \sum_n [H_s(n+1)]^{\frac{1}{2}} |n+1, s\rangle\langle n, s-1|$$

it is possible to factorize H_s as

$$H_s = X(s)_+ \, X(s)_-$$

modulo the omission of the ground state $|0, s\rangle$ (which amounts to substract the corresponding eigenvalue from the spectrum of H_s). Let us now define: (i) the two (supercharge) operators

$$q(s)_- = X(s)_- \, \Pi_s, \quad q(s)_+ = X(s)_+ \, \Pi_{s-1}$$

and (ii) the (Hamiltonian) operator

$$h(s) = X(s)_- \, X(s)_+ \, \Pi_{s-1} + X(s)_+ \, X(s)_- \, \Pi_s. \tag{31}$$

It is then a simple matter of calculation to prove that $h(s)$ is self-adjoint and that

$$q(s)_+ = q(s)_-^\dagger, \quad q(s)_\pm^2 = 0, \quad h(s) = \{q(s)_-, q(s)_+\}, \quad [h(s), q(s)_\pm] = 0.$$

Consequently, the doublet $(h(s), q(s))_2$, with $q(s) \equiv q(s)_-$, satisfies Eq. (18) with $k = 2$ and thus defines an ordinary sypersymmetric quantum-mechanical system (corresponding to $k = 2$).

The Hamiltonian $h(s)$ is amenable to a form more appropriate for discussing the link between ordinary sQM and fractional sQM. Indeed, we can show that

$$X(s)_- \, X(s)_+ = H_s(N+1). \tag{32}$$

Then, by combining Eqs. (2), (30) and (32), Eq. (31) leads to the important relation

$$h(s) = H_{s-1} \, \Pi_{s-1} + H_s \, \Pi_s \tag{33}$$

to be compared with the expansion of H in terms of supersymmetric partners H_s (see Eq. (29)).

As a result, the system $(H, Q)_k$, corresponding to k-fractional sQM, can be described in terms of $k-1$ sub-systems $(h(s), q(s))_2$, corresponding to ordinary sQM. The Hamiltonian $h(s)$ is given as a sum involving the supersymmetric partners H_{s-1} and H_s (see Eq. (33)). Since the supercharges $q(s)_\pm$ commute with the Hamiltonian $h(s)$, it follows that

$$H_{s-1} X(s)_- = X(s)_- H_s, \quad H_s X(s)_+ = X(s)_+ H_{s-1}.$$

As a consequence, the operators $X(s)_+$ and $X(s)_-$ render possible to pass from the spectrum of H_{s-1} and H_s to the one of H_s and H_{s-1}, respectively. This result is quite familiar for ordinary sQM (corresponding to $s = 2$).

For $k = 2$, the operator $h(1)$ is nothing but the total Hamiltonian H corresponding to ordinary sQM. For arbitrary k, the other operators $h(s)$ are simple replicas (except for the ground state of $h(s)$) of $h(1)$. In this sense, fractional sQM of order k can be considered as a set of $k-1$ replicas of ordinary sQM corresponding to $k=2$ and typically described by $(h(s), q(s))_2$. As a further argument, it is to be emphasized that

$$H = q(2)_- \, q(2)_+ + \sum_{s=2}^{k} q(s)_+ \, q(s)_-$$

which can be identified to $h(2)$ for $k=2$.

4 A fractional supersymmetric oscillator

4.1 A special case of W_k

In this section, we deal with the particular case where $f_s = 1$ and the deformed bosons $b(s)_\pm \equiv b_\pm$ are independent of s with $s = 0, 1, \cdots, k-1$. We thus end up with a pair (b_-, b_+) of ordinary bosons, satisfying $[b_-, b_+] = 1$, and a pair (f_-, f_+) of k-fermions. The ordinary bosons $b_\pm$ and the k-fermions $f_\pm$ may be considered as originating from the decomposition of a pair of Q-uons when Q goes to the root of unity q (see Appendix B).

Here, the two operators X_- and X_+ are given by Eqs. (9) and (10), where now $b_\pm$ are ordinary boson operators. They satisfy the commutation relation $[X_-, X_+] = 1$. Then, the number operator N may defined by

$$N = X_+ X_-, \tag{34a}$$

which is amenable to the form

$$N = b_+ b_-. \tag{34b}$$

Finally, the grading operator K is still defined by Eq. (11). We can check that the operators X_-, X_+, N and K so-defined generate the generalized Weyl-Heisenberg algebra W_k defined by Eq. (2) with $f_s = 1$ for $s = 0, 1, \cdots, k-1$. The latter algebra W_k can thus be realized with multilinear forms involving ordinary boson operators $b_\pm$ and k-fermion operators $f_\pm$.

4.2 The resulting fractional supersymmetric oscillator

The supercharge operators Q_- and Q_+ as well as the Hamiltonian H associated with the algebra W_k can be constructed, in terms of the operators b_-, b_+, f_- and f_+, according to the prescriptions given in Section 3.3. This leads to the expression

$$H = (k-1) b_+ b_- + (k-1) \sum_{s=0}^{k-1} (s + 1 - \frac{1}{2} k) \Pi_{k-s}$$

to be compared with Eq. (27).

Most of the properties of the Hamiltonian H are essentially the same as the ones given above for the Hamiltonian (27). In particular, we can write (see Eqs. (29) and (30))

$$H = \sum_{s=1}^{k} H_s \Pi_s, \quad H_s = (k-1)\left(b_+ b_- + \frac{1}{2}k + 1 - s\right)$$

and thus H is a linear combination of projection operators with coefficients H_s corresponding to isospectral Hamiltonians (remember that $\Pi_k = \Pi_0$).

To close this section, let us mention that the fractional supercoherent state $|z, \theta)$ defined in Appendix B is a coherent state corresponding to the Hamiltonian H. As a point of fact, we can check that the action of the evolution operator $\exp(-\mathrm{i}Ht)$ on the state $|z, \theta)$ gives

$$\exp(-\mathrm{i}Ht)\, |z, \theta) = \exp\left[-\frac{\mathrm{i}}{2}(k-1)(k+2)t\right]\, |\mathrm{e}^{-\mathrm{i}(k-1)t} z, \mathrm{e}^{+\mathrm{i}(k-1)t}\theta),$$

i.e., another fractional supercoherent state.

4.3 Examples

4.3.1 Example 1

As a first example, we take $k = 2$, i.e., $q = -1$. Then, the operators

$$X_\pm = b_\pm\, (f_- + f_+)$$

and the operators K and N, see Eqs. (11) and (34), are defined in terms of bilinear forms of ordinary bosons (b_-, b_+) and ordinary fermions (f_-, f_+). The operators X_-, X_+, N and K satisfy

$$[X_-, X_+] = 1, \quad [N, X_\pm] = \pm X_\pm,$$

$$[K, X_\pm]_+ = 0, \quad [K, N] = 0, \quad K^2 = 1,$$

which reflect bosonic and fermionic degrees of freedom, the bosonic degree corresponding to the triplet (X_-, X_+, N) and the fermionic degree to the Klein involution operator K. The projection operators

$$\Pi_0 = \frac{1}{2}(1+K) = 1 - f_+ f_-, \quad \Pi_1 = \frac{1}{2}(1-K) = f_+ f_-$$

are here simple chirality operators and the supercharges

$$Q_- = X_- \Pi_0 = b_- f_+, \quad Q_+ = X_+ \Pi_1 = b_+ f_-$$

have the property

$$Q_-^2 = Q_+^2 = 0.$$

The Hamiltonian H assumes the form

$$H = \{Q_-, Q_+\}$$

which can be rewritten as

$$H = b_+ b_- \Pi_0 + b_- b_+ \Pi_1.$$

It is clear that the operator H is self-adjoint and commutes with Q_- and Q_+. Note that we recover that Q_-, Q_+ and H span the Lie superalgebra $s\ell(1/1)$. We have

$$H = b_+ b_- + f_+ f_-$$

so that H acts on the Z_2-graded space $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1$. The operator H corresponds to the ordinary or Z_2-graded supersymmetric oscillator whose energy spectrum E is (in a symbolic way)

$$E = 1 \oplus 2 \oplus 2 \oplus \cdots$$

with equally spaced levels, the ground state being a singlet (denoted by 1) and all the excited states (viz., an infinite sequence) being doublets (denoted by 2). Finally, note that the fractional supercoherent state $|z, \theta)$ of Appendix B with $k = 2$ is a coherent state for the Hamiltonian H (see also Ref. [53]).

4.3.2 Example 2

We continue with $k = 3$, i.e.,

$$q = \exp\left(\frac{2\pi \mathrm{i}}{3}\right).$$

In this case, we have

$$X_- = b_- \left(f_- + \frac{f_+^2}{[[2]]_q!}\right) = b_- \left(f_- - q f_+^2\right),$$

$$X_+ = b_+ \left(f_- + \frac{f_+^2}{[[2]]_q!}\right)^2 = b_+ \left(f_+ + f_-^2 + q^2 f_+^2 f_-\right).$$

Furthermore, K and N are given by (11) and (34), where here (b_-, b_+) are ordinary bosons and (f_-, f_+) are 3-fermions. We hence have

$$[X_-, X_+] = 1, \quad [N, X_\pm] = \pm X_\pm,$$

$$[K, X_+]_q = [K, X_-]_{\bar{q}} = 0, \quad [K, N] = 0, \quad K^3 = 1.$$

Our general definitions can be specialized to

$$\begin{aligned}
\Pi_0 &= \frac{1}{3}\left(1 + q^3 K + q^3 K^2\right) \\
\Pi_1 &= \frac{1}{3}\left(1 + q^1 K + q^2 K^2\right) \\
\Pi_2 &= \frac{1}{3}\left(1 + q^2 K + q^1 K^2\right)
\end{aligned}$$

or equivalently

$$\begin{aligned}
\Pi_0 &= 1 + (q-1) f_+ f_- - q f_+ f_- f_+ f_- \\
\Pi_1 &= -q f_+ f_- + (1+q) f_+ f_- f_+ f_- \\
\Pi_2 &= f_+ f_- - f_+ f_- f_+ f_-
\end{aligned}$$

for the projection operators and to

$$Q_- = X_-(\Pi_0 + \Pi_2) = b_- f_+ \left(f_-^2 - q f_+ \right)$$

$$Q_+ = X_+(\Pi_1 + \Pi_2) = b_+ \left(f_- - q f_+^2 \right) f_-$$

for the supercharges with the property

$$Q_-^3 = Q_+^3 = 0.$$

By introducing the Hamiltonian H via

$$Q_-^2 Q_+ + Q_- Q_+ Q_- + Q_+ Q_-^2 = Q_- H$$

we obtain

$$H = (2b_+ b_- - 1)\, \Pi_0 + (2b_+ b_- + 3)\, \Pi_1 + (2b_+ b_- + 1)\, \Pi_2$$

which acts on the Z_3-graded space $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1 \oplus \mathcal{F}_2$ and can be rewritten as

$$H = 2b_+ b_- - 1 + 2(1-2q) f_+ f_- + 2(1+2q) f_+ f_- f_+ f_-$$

in terms of boson and 3-fermion operators. We can check that the operator H is self-adjoint and commutes with Q_- and Q_+. The energy spectrum of H reads

$$E = 1 \oplus 2 \oplus 3 \oplus 3 \oplus \cdots .$$

It contains equally spaced levels with a nondegenerate ground state (denoted as 1), a doubly degenerate first excited state (denoted as 2) and an infinite sequence of triply degenerate excited states (denoted as 3).

5 Differential realizations

In this section, we consider the case of the algebra W_k defined by Eqs. (2b)-(2e) and Eq. (14) with $c_0 = 1$ and $c_s = c\delta(s,1)$, $c \in \mathbf{R}$, for $s = 1, 2, \cdots, k-1$. In other words, we have

$$[X_-, X_+] = 1 + cK, \quad K^k = 1, \tag{35a}$$

$$[K, X_+]_q = [K, X_-]_{\bar{q}} = 0, \tag{35b}$$

which corresponds to the C_λ-extended oscillator. The operators X_-, X_+ and K can be realized in terms of a bosonic variable x and its derivative $\frac{d}{dx}$ satisfying

$$[\frac{d}{dx}, x] = 1$$

and a k-fermionic variable (or generalized Grassmann variable) θ and its derivative $\frac{d}{d\theta}$ satisfying [24,36] (see also Refs. [25-31])

$$[\frac{d}{d\theta}, \theta]_{\bar{q}} = 1, \quad \theta^k = \left(\frac{d}{d\theta}\right)^k = 0.$$

Of course, the sets $\{x, \frac{d}{dx}\}$ and $\{\theta, \frac{d}{d\theta}\}$ commute. It is a simple matter of calculation to derive the two following identities

$$\left(\frac{d}{d\theta} + \frac{\theta^{k-1}}{[[k-1]]_{\bar{q}}!}\right)^k = 1$$

and

$$(\frac{d}{d\theta}\theta - \theta\frac{d}{d\theta})^k = 1,$$

which are essential for the realizations given below.

As a first realization, we can show that the shift operators

$$X_- = \frac{d}{dx}\left(\frac{d}{d\theta} + \frac{\theta^{k-1}}{[[k-1]]_{\bar{q}}!}\right)^{k-1} - \frac{c}{x}\theta,$$

$$X_+ = x\left(\frac{d}{d\theta} + \frac{\theta^{k-1}}{[[k-1]]_{\bar{q}}!}\right),$$

and the Witten grading operator

$$K = [\frac{d}{d\theta}, \theta]$$

satisfy Eq. (35). This realization of X_-, X_+ and K clearly exibits the bosonic and k-fermionic degrees of freedom via the sets $\{x, \frac{d}{dx}\}$ and $\{\theta, \frac{d}{d\theta}\}$, respectively. In the particular case $k = 2$, the k-fermionic variable θ is an ordinary Grassmann variable and the supercharge operators Q_- and Q_+ take the simple form

$$Q_- = \left(\frac{d}{dx} - \frac{c}{x}\right)\theta, \tag{36a}$$

$$Q_+ = x\frac{d}{d\theta}. \tag{36b}$$

(Note that the latter realization for Q_- and Q_+ is valid for $k = 3$ too.)

Another possible realization of X_- and X_+ for arbitrary k is

$$X_- = P\left(\frac{d}{d\theta} + \frac{\theta^{k-1}}{[[k-1]]_{\bar{q}}!}\right)^{k-1} - \frac{c}{x}\theta,$$

$$X_+ = X\left(\frac{d}{d\theta} + \frac{\theta^{k-1}}{[[k-1]]_{\bar{q}}!}\right),$$

where P and X are the two canonically conjugated quantities

$$P = \frac{1}{\sqrt{2}} \left(x + \frac{d}{dx} - \frac{c}{2x} K \right)$$

and

$$X = \frac{1}{\sqrt{2}} \left(x - \frac{d}{dx} + \frac{c}{2x} K \right).$$

This realization is more convenient for a Schrödinger type approach to the supersymmetric Hamiltonian H. According to Eq. (23), we can derive an Hamiltonian H involving bosonic and k-fermionic degrees of freedom. To illustrate this point, let us continue with the particular case $k = 2$. It can be seen that the supercharge operators (36) must be replaced by

$$Q_- = \left(P - \frac{c}{X} \right) \theta,$$

$$Q_+ = X \frac{d}{d\theta}.$$

(Note the formal character of Q_- since the definition of Q_- lies on the existence of an inverse for the operator X.) Then, we obtain the following Hamiltonian

$$H = -\frac{1}{2} \left[\left(\frac{d}{dx} - \frac{c}{2x} K \right)^2 - x^2 + K + c(1 + K) \right].$$

For $c = 0$, we have (cf. Ref. [8])

$$H = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 - \frac{1}{2} K$$

that is the Hamiltonian for an ordinary super-oscillator, i.e., a Z_2-graded supersymmetric oscillator. Here, the bosonic character arises from the bosonic variable x and the fermionic character from the ordinary Grassmann variable θ in K.

6 Closing remarks

The basic ingredient for the present work is the definition of a generalized Weyl-Heisenberg algebra W_k that depends on k structure constants f_s ($s = 0, 1, \cdots, k - 1$). We have shown how to construct $\mathcal{N} = 2$ fractional supersymmetric Quantum Mechanics of order k, $k \in \{0, 1\}$, by means of this Z_k-graded algebra W_k. The $\mathcal{N} = 2$ dependent supercharges and a general Hamiltonian are derived in terms of the generators of W_k. This general fractional supersymmetric Hamiltonian is a linear combination of isospectral supersymmetric partners $H_0, H_{k-1}, \cdots, H_1$ and this result is at the root of the development of fractional supersymmetric Quantum Mechanics of order k as a set of replicas of ordinary supersymmetric Quantum Mechanics (corresponding to $k = 2$). The general Hamiltonian covers various dynamical systems corresponding to translational and cyclic shape-invariant potentials.

A special attention has been given to the fractional supersymmetric oscillator. From a general point of view, the formalism presented in this paper is useful for studying exact integrable quantum systems and for constructing their coherent states.

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Appendix A: Connection between W_k and $U_q(sl_2)$

Let us now show that the quantum algebra $U_q(sl_2)$, with q being the k-th root of unity given by (1), turns out to be a particular form of W_k. The algebra $U_q(sl_2)$ is spanned by the generators J_-, J_+, q^{J_3} and q^{-J_3} that satisfy the relationships

$$[J_+, J_-] = [2J_3]_q,$$

$$q^{J_3} J_+ q^{-J_3} = qJ_+, \quad q^{J_3} J_- q^{-J_3} = \bar{q}J_-,$$

$$q^{J_3} q^{-J_3} = q^{-J_3} q^{J_3} = 1.$$

It is straightforward to prove that the operator

$$J^2 = J_- J_+ + \frac{q^{+1}q^{2J_3} + q^{-1}q^{-2J_3}}{(q - q^{-1})^2}$$

or

$$J^2 = J_+ J_- + \frac{q^{-1}q^{2J_3} + q^{+1}q^{-2J_3}}{(q - q^{-1})^2}$$

is an invariant of $U_q(sl_2)$. In view of Eq. (1), the operators J_-^k, J_+^k, $(q^{J_3})^k$, and $(q^{-J_3})^k$ belong, likewise J^2, to the center of $U_q(sl_2)$.

In the case where the deformation parameter q is a root of unity, the representation theory of $U_q(sl_2)$ is richer than the one for q generic. The algebra $U_q(sl_2)$ admits finite-dimensional representations of dimension k such that

$$J_-^k = A, \quad J_+^k = B,$$

where A and B are constant matrices. Three types of representations have been studied in the literature [49]:

(i) $A = B = 0$ (nilpotent representations),
(ii) $A = B = 1$ (cyclic or periodic representations),
(iii) $A = 0$ and $B = 1$ or $A = 1$ and $B = 0$ (semi-periodic representations).
Indeed, the realization of fractional sQM based on $U_q(sl_2)$ does not depend of the choice (i), (ii) or (iii) in contrast with the work in Ref. [36] where nilpotent representations corresponding to the choice (i) were considered. The only important ingredient is to take

$$\left(q^{J_3}\right)^k = 1$$

that ensures a Z_k-grading of the Hilbertean representation space of $U_q(sl_2)$.

The contact with the algebra W_k is established by putting

$$X_\pm = J_\pm, \quad N = J_3, \quad K = q^{J_3},$$

and by using the definition (3) of Π_s as function of K. Here, the operator Π_s is a projection operator on the subspace, of the representation space of $U_q(sl_2)$, corresponding to a given eigenvalue of J_3. It is easy to check that the operators X_-, X_+, N and K satisfy Eq. (2) with

$$f_s(N) = -[2s]_q = -\frac{\sin \frac{4\pi s}{k}}{\sin \frac{2\pi}{k}}$$

for $s = 0, 1, \cdots, k-1$. The quantum algebra $U_q(sl_2)$, with q given by (1), then appears as a further particular case of the generalized Weyl-Heisenberg algebra W_k.

Appendix B: A Q-uon $\to$ boson + k-fermion decomposition

We shall limit ourselves to give an outline of this decomposition (see Dunne *et al.* [54] and Mansour *et al.* [55] for an alternative and more rigorous mathematical presentation based on the isomorphism between the braided Z-line and the (z, θ)-superspace). We start from a Q-uon algebra spanned by three operators a_-, a_+ and N_a satisfying the relationships [18] (see also Refs. [19-22])

$$[a_-, a_+]_Q = 1, \quad [N_a, a_\pm] = \pm a_\pm,$$

where Q is generic (a real number different from zero). The action of the operators a_-, $a_+ = a_-^\dagger$ and $N_a = N_a^\dagger$ on a Fock space $\mathcal{F} = \{|n\rangle : n \in \mathbf{N}\}$ is given by

$$N_a|n\rangle = n|n\rangle,$$

and

$$a_-|n\rangle = ([[n + \sigma - \frac{1}{2}]]_Q)^\alpha \, |n-1\rangle,$$

$$a_+|n\rangle = ([[n + \sigma + \frac{1}{2}]]_Q)^\beta \, |n+1\rangle,$$

where $\alpha + \beta = 1$ with $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$. For $\alpha = \beta = \frac{1}{2}$, let us consider the Q-deformed Glauber coherent state [18] (see also Ref. [56])

$$|Z) = \sum_{n=0}^{\infty} \frac{(Za_+)^n}{[[n]]_Q!} |0\rangle = \sum_{n=0}^{\infty} \frac{Z^n}{([[n]]_Q!)^{\frac{1}{2}}} |n\rangle$$

(with $Z \in \mathbf{C}$). If we do the replacement

$$Q \rightsquigarrow q = \exp\left(\frac{2\pi \mathrm{i}}{k}\right), \quad k \in \mathbf{N} \setminus \{0, 1\},$$

then we have $[[k]]_Q! \to [[k]]_q! = 0$. Therefore, in order to give a sense to $|Z)$ for $Q \rightsquigarrow q$, we have to do the replacement

$$a_+ \rightsquigarrow f_+ \quad \text{with} \quad f_+^k = 0,$$

$$a_- \rightsquigarrow f_- \quad \text{with} \quad f_-^k = 0.$$

We thus end up with what we call a k-fermionic algebra F_k spanned by the operators f_-, f_+ and $N_f \equiv N_a$ completed by the adjoints $f_+^\dagger$ and $f_-^\dagger$ of f_+ and f_-, respectively [31,33]. The defining relations for the k-fermionic algebra F_k are

$$[f_-, f_+]_q = 1, \quad [N_f, f_\pm] = \pm f_\pm, \quad f_-^k = f_+^k = 0,$$

$$[f_+^\dagger, f_-^\dagger]_{\bar{q}} = 1, \quad [N_f, f_\pm^\dagger] = \mp f_\pm^\dagger, \quad \left(f_-^\dagger\right)^k = \left(f_+^\dagger\right)^k = 0,$$

$$f_- f_+^\dagger - q^{-\frac{1}{2}} f_+^\dagger f_- = 0, \quad f_+ f_-^\dagger - q^{+\frac{1}{2}} f_-^\dagger f_+ = 0.$$

The case $k = 2$ corresponds to ordinary fermion operators and the case $k \to \infty$ to ordinary boson operators. In the two latter cases, we can take $f_- \equiv f_+^\dagger$ and $f_+ \equiv f_-^\dagger$; in the other cases, the consideration of the two couples $(f_-, f_+^\dagger)$ and $(f_+, f_-^\dagger)$ is absolutely necessary. In the case where k is arbitrary, we shall speak of k-fermions. The k-fermions are objects interpolating between fermions and bosons. They share some properties with the para-fermions [24,25,27] and the anyons as introduced by Goldin *et al.* [10] (see also Ref. [9]). If we define

$$b_\pm = \lim_{Q \rightsquigarrow q} \frac{a_\pm^k}{([[k]]_Q!)^{\frac{1}{2}}}$$

we obtain

$$[b_-, b_+] = 1$$

so that the operators b_- and b_+ can be considered as ordinary boson operators. This is at the root of the two following results [31].

As a first result, the set $\{a_-, a_+\}$ gives rise, for $Q \rightsquigarrow q$, to two commuting sets: The set $\{b_-, b_+\}$ of boson operators and the set of k-fermion operators $\{f_-, f_+\}$.

As a second result, this decomposition leads to the replacement of the Q-deformed coherent state $|Z)$ by the so-called fractional supercoherent state

$$|z,\theta) = \sum_{n=0}^{\infty}\sum_{s=0}^{k-1} \frac{\theta^s}{([[s]]_q!)^{\frac{1}{2}}} \frac{z^n}{\sqrt{n!}} |n,s\rangle,$$

where z is a (bosonic) complex variable and θ a (k-fermionic) generalized Grassmann variable [24,27,36,57] with $\theta^k = 0$. The fractional supercoherent state $|z,\theta)$ is an eigenvector of the product $f_- b_-$ with the eigenvalue $z\theta$. The state $|z^k,\theta)$ can be seen to be a linear combination of the coherent states introduced by Vourdas [58] with coefficients in the generalized Grassmann algebra spanned by θ and the derivative $\frac{d}{d\theta}$.

In the case $k = 2$, the state $|z,\theta)$ turns out to be a coherent state for the ordinary (or Z_2-graded) supersymmetric oscillator [53]. For $k \geq 3$, the state $|z,\theta)$ is a coherent state for the Z_k-graded supersymmetric oscillator (see Section 4).

It is possible to find a realization of the operators f_-, f_+, $f_+^\dagger$ and $f_-^\dagger$ in terms of Grassmann variables $(\theta,\bar{\theta})$ and their q- and $\bar{q}$-derivatives $(\partial_\theta,\partial_{\bar{\theta}})$. We take Grassmann variables θ and $\bar{\theta}$ such that $\theta^k = \bar{\theta}^k = 0$ [24,27,36,57]. The sets $\{1,\theta,\cdots,\theta^{k-1}\}$ and $\{1,\bar{\theta},\cdots,\bar{\theta}^{k-1}\}$ span the same Grassmann algebra Σ_k. The q- and $\bar{q}$-derivatives are formally defined by

$$\begin{aligned} \partial_\theta f(\theta) &= \frac{f(q\theta)-f(\theta)}{(q-1)\theta}, \\ \partial_{\bar{\theta}} g(\bar{\theta}) &= \frac{g(\bar{q}\bar{\theta})-g(\bar{\theta})}{(\bar{q}-1)\bar{\theta}}. \end{aligned}$$

Therefore, by taking

$$f_+ = \theta,\ f_- = \partial_\theta,\ f_-^\dagger = \bar{\theta},\ f_+^\dagger = \partial_{\bar{\theta}},$$

we have

$$\partial_\theta\theta - q\theta\partial_\theta = 1,\ (\partial_\theta)^k = \theta^k = 0,$$

$$\partial_{\bar{\theta}}\bar{\theta} - \bar{q}\bar{\theta}\partial_{\bar{\theta}} = 1,\ (\partial_{\bar{\theta}})^k = \bar{\theta}^k = 0,$$

$$\partial_\theta\partial_{\bar{\theta}} - q^{-\frac{1}{2}}\partial_{\bar{\theta}}\partial_\theta = 0,\ \theta\bar{\theta} - q^{+\frac{1}{2}}\bar{\theta}\theta = 0.$$

Following Majid and Rodríguez-Plaza [57], we define the integration process

$$\int d\theta\,\theta^n = \int d\bar{\theta}\,\bar{\theta}^n = 0 \text{ for } n = 0,1,\cdots,k-2$$

and

$$\int d\theta\,\theta^{k-1} = \int d\bar{\theta}\,\bar{\theta}^{k-1} = 1$$

which gives the Berezin integration for the particular case $k = 2$.

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